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NATURAL FREQUENCIES OF A LINEAR ELASTICA CARRYING ANY NUMBER OF SPRUNG MASSES

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1. INTRODUCTION

The vibration of a linear elastica carrying any number of sprung masses has received considerable interest in recent years, and has been studied by many authors [1–8]. While in theory most of the proposed formalisms may be extended to solve for the natural frequencies of a linear structure carrying any number of spring–mass systems, their actual implementation may be difficult because of the associated mathematical complexity. Thus, most examples used to demonstrate the feasibility of the various approaches have been limited to only one sprung mass.

Recently, Gürgöze [9] presented two approaches to compute the natural frequencies of a Euler–Bernoulli beam to which several spring–mass systems are attached in span. He first used the Lagrange multipliers formalism introduced by Dowell [1] to obtain the characteristic determinant, which can then be used to compute the natural frequencies of the combined structure. While the results are concise, the inherent nature of the formalism misses certain natural frequencies when the spring–mass systems are located at the nodes of the beam. In addition, the Lagrange multipliers approach can be fairly laborious to apply, because one needs to introduce S Lagrange multipliers and to formulate S constraint equations, where S is the number of sprung masses. Gürgöze then used the assumed-modes method [10], in addition to a co-ordinate transformation, to obtain yet another characteristic determinant that can be used to solve for the natural frequencies. The latter approach, while simple, leads to a characteristic determinant of size $(N + S) \times (N + S)$, where N represents the number of modes used in the assumed modes expansion.

Wu and Chou [11] introduced a numerical technique to obtain the exact solutions for the lower modes of vibration of a uniform beam carrying any number of sprung masses with various boundary conditions. They determined the coefficient matrix for a beam element, and used the conventional assembly technique for the finite-element method to determine the overall coefficient matrix, from which they solved for the natural frequencies and mode shapes of the entire structure. While their approach leads to the exact solution, their scheme is highly numerically intensive, since a characteristic determinant of size $5S + 4$ needs to be solved. Moreover, the element coefficient matrix that they derived is strictly valid for a beam. Thus, their results cannot be easily extended to other linear structures, and more work is needed before their scheme can be applied to other types of elastica.

In this technical note, the discretized governing equations for a linear elastica carrying a number of sprung masses (see Figure 1) are first obtained by using the common assumed-modes method. Using this approach, the natural frequencies squared correspond

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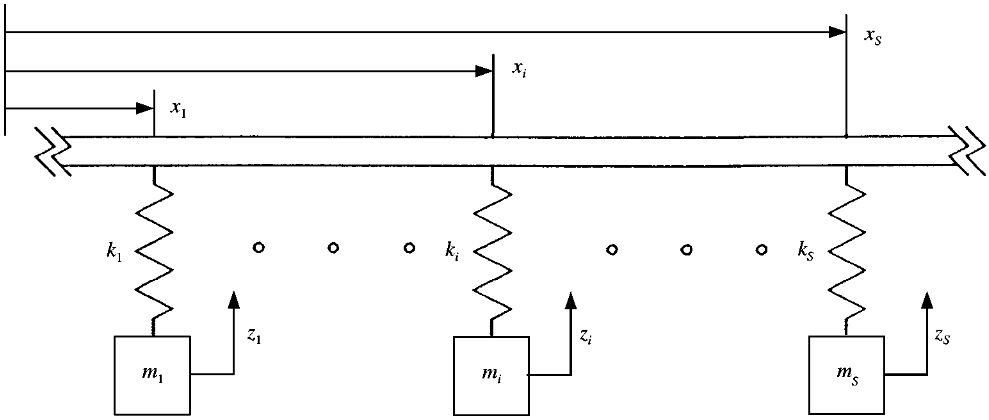

Figure 1. An arbitrarily supported, linear elastic structure carrying any number of sprung masses.

to the eigenvalues of a generalized eigenvalue problem. By manipulating the characteristic determinant associated with the generalized eigenvalue problem, it can be algebraically reduced to one of a smaller size, thus providing an alternative means to solve for the natural frequencies of the combined system. The advantages of the proposed scheme will be discussed and highlighted, and numerical examples will be provided to illustrate the utility of the new formalism.

2. THEORY

Consider the free vibration of an arbitrarily supported, linear structure carrying S-sprung masses as shown in Figure 1. Using the assumed-modes method, the physical deflection of the structure at a point x is given by

$$w(x, t) = \sum_{i=1}^{N} \phi_i(x)\eta_i(t), \tag{1}$$

where the $\phi_i(x)$ are the eigenfunctions of the linear structure (the elastica without any sprung masses) that serve as the basis functions for this approximate solution, the $\eta_i(t)$ are the corresponding generalized co-ordinates, and N is the number of modes used in the assumed-modes expansion. The total kinetic energy of the combined system is given by

$$T = \frac{1}{2}\sum_{i=1}^{N} M_i\dot{\eta}_i^2(t) + \frac{1}{2}\sum_{i=1}^{S} m_i\dot{z}_i^2(t), \tag{2}$$

where the M_i are the generalized masses, m_i is the mass of the ith oscillator, $z_i(t)$ is its displacement, S is the total number of sprung masses attached to the elastica, and an overdot denotes a derivative with respect to time. The total potential energy is given by

$$V = \frac{1}{2}\sum_{i=1}^{N} K_i\eta_i^2(t) + \frac{1}{2}\sum_{i=1}^{S} k_i[z_i(t) - w(x_i, t)]^2, \tag{3}$$

where the K_i are the generalized spring constants, k_i is the spring stiffness of the ith oscillator, and $w(x_i, t)$ represents the lateral displacement of the beam at x_i.

Applying Lagrange's equations and assuming simple harmonic motion,

$$\eta_i(t) = \bar{\eta}_i e^{j\omega t}, \qquad z_i(t) = \bar{z}_i e^{j\omega t}, \tag{4}$$

where $j = \sqrt{-1}$ and ω is the natural frequency, the frequency equation for the system of Figure 1 can be obtained by solving the following generalized eigenvalue problem:

$$\begin{bmatrix} [\mathscr{K}] & [R] \\ [R]^T & [k] \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{\mathbf{z}} \end{bmatrix} = \omega^2 \begin{bmatrix} [\mathscr{M}] & [0] \\ [0]^T & [m] \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{\mathbf{z}} \end{bmatrix}, \tag{5}$$

where $\bar{\boldsymbol{\eta}} = [\bar{\eta}_1 \;\; \bar{\eta}_2 \cdots \bar{\eta}_N]^T$ is the vector of generalized co-ordinates, $\bar{\mathbf{z}} = [\bar{z}_1 \;\; \bar{z}_2 \cdots \bar{z}_S]^T$ is the vector of mass displacements, and the $S \times S$ matrices $[m]$ and $[k]$ are both diagonal of the form

$$[m] = \text{diag}\,[m_i], \qquad [k] = \text{diag}\,[k_i]. \tag{6}$$

The $N \times N$ $[\mathscr{M}]$ and $[\mathscr{K}]$ matrices of equation (5) are

$$[\mathscr{M}] = [M^d], \qquad [\mathscr{K}] = [K^d] + \sum_{i=1}^{S} k_i \boldsymbol{\phi}(x_i)\boldsymbol{\phi}^T(x_i), \tag{7}$$

where $[M^d]$ and $[K^d]$ are diagonal matrices whose ith diagonal elements are M_i and K_i,

$$\boldsymbol{\phi}(x_i) = [\phi_1(x_i) \;\; \phi_2(x_i) \;\cdots\; \phi_N(x_i)]^T, \tag{8}$$

and the $N \times S$ matrix $[R]$ is given by

$$[R] = [-k_1\boldsymbol{\phi}(x_1) \;\cdots\; -k_i\boldsymbol{\phi}(x_i) \;\cdots\; -k_S\boldsymbol{\phi}(x_S)]. \tag{9}$$

Thus, $[\mathscr{M}]$ is a diagonal matrix and $[\mathscr{K}]$ is a diagonal matrix modified by S rank one matrices. Once the linear elastic structure and the attachment locations are specified, and the spring–mass parameters are given, the natural frequencies of the combined system can be readily obtained by solving the generalized eigenvalue problem of equation (5).

At first glance, it appears that a frequency analysis for the system of Figure 1 requires the solution of a generalized eigenvalue problem of size $(N+S)\times(N+S)$ (see equation (5)). However, by simple algebraic manipulation, the generalized eigenvalue problem (5) can be reduced to one of smaller size. From equation (5), one obtains the following expressions for the $\bar{z}_i$:

$$\bar{z}_i = \frac{k_i \boldsymbol{\phi}^T(x_i)}{k_i - \omega^2 m_i}\bar{\boldsymbol{\eta}}, \quad i = 1, 2, \ldots, S. \tag{10}$$

Substituting the expressions of equation (10) into equation (5), the following generalized eigenvalue problem, of size $N \times N$, is obtained:

$$\left\{[K^d] + \sum_{i=1}^{S} \sigma_i \boldsymbol{\phi}(x_i)\boldsymbol{\phi}^T(x_i)\right\}\bar{\boldsymbol{\eta}} = \omega^2 [M^d]\bar{\boldsymbol{\eta}}, \tag{11}$$

where

$$\sigma_i = \frac{k_i m_i \omega^2}{\omega^2 m_i - k_i}. \tag{12}$$

Expanding equation (11), the natural frequencies of the system are given by the solution of the characteristic determinant

$$\det\left\{[K^d] - \omega^2[M^d] + \sum_{i=1}^{S} \sigma_i \boldsymbol{\phi}(x_i)\boldsymbol{\phi}^{\mathrm{T}}(x_i)\right\} = 0, \tag{13}$$

which can be shown [12] to be identical to

$$\det\{[K^d] - \omega^2[M^d]\}\det[B] = \left\{\prod_{i=1}^{N}(K_i - \omega^2 M_i)\right\}\det[B] = 0, \tag{14}$$

where K_i and M_i represent the ith element of $[K^d]$ and $[M^d]$, and the (i, j)th element of $[B]$, of size $S \times S$, is given by

$$b_{ij} = \sum_{r=1}^{N} \frac{\phi_r(x_i)\phi_r(x_j)}{K_r - \omega^2 M_r} + \frac{1}{\sigma_i}\delta_i^j, \quad i, j = 1, 2, \dots, S, \tag{15}$$

and δ_i^j represents the Kronecker delta. When $\omega^2 \neq K_i/M_i$, equation (14) reduces to

$$\det[B] = 0, \tag{16}$$

the same result as equation (25) that Gürgöze [9] obtained by using the Lagrange multipliers formalism.

Comparing equations (14) and (16), one immediately notices the absence of the product terms. These product terms are important, for they serve as a remainder that when the attachment locations for the sprung masses coincide with the nodes of any component mode, $\phi_i(x)$, some of the natural frequencies of the combined system will be identical to those of the linear structure. In this case, equation (14) must be used since equation (16) fails to generate all of the natural frequencies of the combined system. Dowell [13] remedied this difficulty by artificially disassembling the structure, which then allowed him to recover the missing natural frequencies. Using equation (14), on the other hand, one obtains all of the natural frequencies of a linear elastica carrying any number of sprung masses, independent of the attachment locations. Additionally, using the present scheme, one eliminates the mathematical complexity associated with the application of the Lagrange multipliers approach.

Equation (14) also has certain conceptual advantages over equation (5). Specifically, it can be easily modified to compute the natural frequencies of a linear structure carrying S lumped masses or attached to S grounded elastic supports. When k_i tends to infinity, then the σ_i of equation (12) reduces to

$$\sigma_i = -\, m_i\omega^2 \tag{17}$$

and the system of Figure 1 simplifies to a linear elastica with S rigidly attached masses. Similarly, when m_i approaches infinity, the σ_i of equation (12) reduces to

$$\sigma_i = k_i \tag{18}$$

and the system of Figure 1 simplifies to a linear structure attached to S grounded springs. It should be noted that the eigenvalue problems to these systems cannot be easily extracted from equation (5).

3. RESULTS

The proposed scheme of determining the natural frequencies of a linear elastica carrying any number of sprung masses offers numerous advantages. Firstly, equation (14) is simple to code. Given the eigenfunctions, $\phi_i(x)$, of the linear elastica, the parameters for the spring–mass systems, m_i and k_i, and the attachment locations, x_i, equation (14) can be easily programmed. Secondly, it can be extended to accommodate any linear elastic structure with any arbitrary boundary conditions by simply using the appropriate eigenfunctions. Finally, equation (14) can be easily modified to analyze a linear structure with any number of rigidly attached masses or any number of grounded springs. To show the utility of the proposed scheme, the natural frequencies of a uniform fixed–free and simply supported Euler–Bernoulli beam carrying 1, 3 and 5 sprung masses are computed, and the results are compared with published values. In all of the subsequent numerical examples, a double precision version of the CMLIB [14] routine *zeroin* was used to find the roots of the characteristic determinant.

When the beam is fixed–free, let $\phi_i(x)$ be the normalized (with respect to the mass per unit nlength, ρ, of the beam) eigenfunctions of a uniform fixed–free beam given by

$$\phi_i(x) = \frac{1}{\sqrt{\rho L}}\left(\cos\beta_i x - \cosh\beta_i x + \frac{\sin\beta_i L - \sinh\beta_i L}{\cos\beta_i L + \cosh\beta_i L}(\sin\beta_i x - \sinh\beta_i x)\right) \quad (19)$$

such that the generalized masses and stiffnesses of the beam are

$$M_i = 1 \qquad \text{and} \qquad K_i = (\beta_i L)^4 EI/(\rho L^4), \quad (20)$$

where E is the Young's modulus, I is the moment of inertia of the cross-section of the beam, and $\beta_i L$ satisfies the transcendental equation

$$\cos\beta_i L \cosh\beta_i L = -1. \quad (21)$$

When the beam is simply supported, let $\phi_i(x)$ be the normalized eigenfunctions of a uniform simply supported beam given by

$$\phi_i(x) = \sqrt{\frac{2}{\rho L}}\sin\frac{\mathrm{i}\pi x}{L} \quad (22)$$

such that the generalized masses and stiffnesses of the beam become

$$M_i = 1 \qquad \text{and} \qquad K_i = (\mathrm{i}\pi)^4 EI/(\rho L^4). \quad (23)$$

Consider first the case of a uniform Euler–Bernoulli beam carrying one spring–mass system (of stiffness k_1 and mass m_1) at x_1, in which case equation (14) reduces to the simple frequency equation

$$\prod_{i=1}^{N}(K_i - \omega^2 M_i)\left(1 + \frac{k_1 m_1\omega^2}{m_1\omega^2 - k_1}\sum_{r=1}^{N}\frac{\phi_i^2(x_1)}{K_i - \omega^2 M_i}\right) = 0. \quad (24)$$

When the attachment location, x_1, does not coincide with any nodes of the eigenfunctions of the beam, equation (24) simplifies to

$$1 + \frac{k_1 m_1\omega^2}{m_1\omega^2 - k_1}\sum_{r=1}^{N}\frac{\phi_i^2(x_1)}{K_i - \omega^2 M_i} = 0, \quad (25)$$

which corresponds to equation (7a) that Dowell derived in reference [1].

When the beam is cantilevered, $\phi_i(x)$ is given by equation (19), and the natural frequencies can be obtained by solving either equation (24) or equation (25), depending on the attachment location. Table 1 compares the first five natural frequencies obtained by using the proposed formalism, for $x_1 = 0{\cdot}75L$, $k_1 = 3{\cdot}0EI/L^3$ and $m_1 = 0{\cdot}2\rho L$, and those given in reference [11]. Since the attachment location, $x_1 = 0{\cdot}75L$, does not coincide with the node of any of the component modes of a uniform fixed–free beam, equation (25) was used to extract the natural frequencies of the system. From Table 1, note the excellent agreement between the results of equation (25), for $N = 5$, and the solution given in reference [11]. For $N = 14$, the agreement between the two becomes even better.

When the beam is simply supported, $\phi_i(x)$ is given by equation (22), and the natural frequencies can be obtained by solving either equation (24) or equation (25), depending again on the attachment location. Table 2 compares the results given in reference [11] and those obtained by using the proposed approach, for the same set of system parameters as those listed in Table 1. Because $x_1 = 0{\cdot}75L$ coincides with a node of the fourth eigenfunction of a simply supported beam, equation (24) was used to extract the natural frequencies, since the fourth natural frequency of a simply supported beam will coincide with a natural frequency of the combined structure. Dowell [1] noted that if a spring–mass system (which by itself has a rigid body degree of freedom) is attached to another system, a new natural frequency appears between the original pair of frequencies nearest the oscillator frequency. Thus, as expected, for the spring–mass parameters chosen, ω_5 of the combined system coincides with the fourth natural frequency of a simply supported beam, $(\omega_4)_{\text{beam}} = (4\pi)^2 \sqrt{EI/\rho L^4}$. Table 2 shows that the agreement between the results of equation (24), for $N = 5$, and the solution given in reference [11] is excellent. Like before, the agreement between the two is improved for $N = 14$.

TABLE 1

The first five natural frequencies of a fixed–free, uniform Euler–Bernoulli beam carrying one sprung mass at $x_1 = 0{\cdot}75L$. The mass and spring stiffness are $m_1 = 0{\cdot}2\rho L$ and $k_1 = 3{\cdot}0EI/L^3$, respectively, where $EI/L^3 = 6{\cdot}34761 \times 10^4$ N/m and $\rho L = 15{\cdot}3875$ kg

Nat. Freq. (rad/s)	Reference [11]	Present ($N = 5$)	Present ($N = 14$)
ω_1	174·2030	174·2047	174·2030
ω_2	322·1513	322·1596	322·1518
ω_3	1415·5524	1415·5526	1415·5526
ω_4	3964·7796	3964·7798	3964·7797
ω_5	7766·4614	7766·4617	7766·4616

TABLE 2

The first five natural frequencies of a simply supported, uniform Euler–Bernoulli beam carrying one sprung mass at $x_1 = 0{\cdot}75L$. The system parameters are identical to those of Table 1

Nat. Freq. (rad/s)	Reference [11]	Present ($N = 5$)	Present ($N = 14$)
ω_1	243·8579	243·8671	243·8582
ω_2	645·2030	645·2040	645·2030
ω_3	2540·5306	2540·5311	2540·5307
ω_4	5706·1886	5706·1888	5706·1887
ω_5	10142·4012	10142·4018	10142·4018

Consider now the case where multiple sprung masses are attached to a uniform Euler–Bernoulli beam. Because of its simplicity, the characteristic determinant of equation (14) can be easily coded to handle multiple spring–mass systems. Tables 3 and 4 compare the first five natural frequencies of fixed–free and simply supported beams, respectively, carrying three sprung masses, obtained by using the proposed formalism and that given in reference [11], for $x_1 = 0{\cdot}1L$, $k_1 = 3{\cdot}0EI/L^3$, $m_1 = 0{\cdot}2\rho L$; $x_2 = 0{\cdot}4L$, $k_2 = 4{\cdot}5EI/L^3$, $m_2 = 0{\cdot}5\rho L$; $x_3 = 0{\cdot}8L$, $k_3 = 6{\cdot}0EI/L^3$, $m_3 = 1{\cdot}0\rho L$. Because the attachment locations do not coincide with the nodes of any of the component modes of a uniform fixed–free beam, equation (16) was used to extract the natural frequencies of the system. From Tables 3 and 4, note the excellent agreement between the results of equation (16), for $N = 5$, and the solution given in reference [11]. The agreement between the two results becomes even better as N is increased to 14.

Tables 5 and 6 show the first five natural frequencies of fixed–free and simply supported beams, respectively, carrying five sprung masses, obtained by using the proposed scheme and that outlined in reference [11], for $x_1 = 0{\cdot}1L$, $k_1 = 3{\cdot}0EI/L^3$, $m_1 = 0{\cdot}2\rho L$; $x_2 = 0{\cdot}2L$, $k_2 = 3{\cdot}5EI/L^3$, $m_2 = 0{\cdot}3\rho L$; $x_3 = 0{\cdot}4L$, $k_3 = 4{\cdot}5EI/L^3$, $m_3 = 0{\cdot}5\rho L$; $x_4 = 0{\cdot}6L$, $k_4 = 5{\cdot}0EI/L^3$, $m_4 = 0{\cdot}65\rho L$; $x_5 = 0{\cdot}8L$, $k_5 = 6{\cdot}0EI/L^3$, $m_5 = 1{\cdot}0\rho L$. Like before, the attachment locations are distinct from the nodes of any of the component modes of a uniform simply supported beam. Thus, equation (16) was used to determine the natural frequencies of the system. From Tables 5 and 6, note the excellent agreement between the results of equation (16), for $N = 5$, and those given in reference [11]. As N is increased to 14, the agreement between the two results is improved.

As a last example, consider a fixed–free uniform Euler–Bernoulli beam to which two grounded springs and one lumped mass are attached as shown in Figure 2, for $x_1 = 0{\cdot}3L$,

TABLE 3

The first five natural frequencies of a fixed–free, uniform Euler–Bernoulli beam carrying three sprung masses, for $x_1 = 0{\cdot}1L$, $k_1 = 3{\cdot}0EI/L^3$, $m_1 = 0{\cdot}2\rho L$; $x_2 = 0{\cdot}4L$, $k_2 = 4{\cdot}5EI/L^3$, $m_2 = 0{\cdot}5\rho L$; $x_3 = 0{\cdot}8L$, $k_3 = 6{\cdot}0EI/L^3$, $m_3 = 1{\cdot}0\rho L$. *The system parameters are* $EI/L^3 = 6{\cdot}34761 \times 10^4$ N/m *and* $\rho L = 15{\cdot}3875$ kg

Nat. Freq. (rad/s)	Reference [11]	Present ($N = 5$)	Present ($N = 14$)
ω_1	102·7944	102·7962	102·7946
ω_2	188·7347	188·7433	188·7352
ω_3	248·6439	248·6613	248·6443
ω_4	349·1161	349·1262	349·1170
ω_5	1427·9521	1427·9535	1427·9523

TABLE 4

The first five natural frequencies of a simply supported, uniform Euler–Bernoulli beam carrying three sprung masses. The system parameters are identical to those of Table 3

Nat. Freq. (rad/s)	Reference [11]	Present ($N = 5$)	Present ($N = 14$)
ω_1	152·7341	152·7438	152·7345
ω_2	185·0950	185·1045	185·0954
ω_3	247·8314	247·8397	247·8319
ω_4	677·5961	677·6032	677·5963
ω_5	2548·6577	2548·6593	2548·6579

TABLE 5

The first five natural frequencies of a fixed–free, uniform Euler–Bernoulli beam carrying five sprung masses, for $x_1 = 0{\cdot}1L$, $k_1 = 3{\cdot}0EI/L^3$, $m_1 = 0{\cdot}2\rho L$; $x_2 = 0{\cdot}2L$, $k_2 = 3{\cdot}5EI/L^3$, $m_2 = 0{\cdot}3\rho L$; $x_3 = 0{\cdot}4L$, $k_3 = 4{\cdot}5EI/L^3$, $m_3 = 0{\cdot}5\rho L$; $x_4 = 0{\cdot}6L$, $k_4 = 5{\cdot}0EI/L^3$, $m_4 = 0{\cdot}65\rho L$; $x_5 = 0{\cdot}8L$, $k_5 = 6{\cdot}0EI/L^3$, $m_5 = 1{\cdot}0\rho L$. *The system parameters are* $EI/L^3 = 6{\cdot}34761 \times 10^4$ N/m *and* $\rho L = 15{\cdot}3875$ kg

Nat. Freq. (rad/s)	Reference [11]	Present ($N = 5$)	Present ($N = 14$)
ω_1	97·4880	97·4893	97·4881
ω_2	171·6770	171·6857	171·6776
ω_3	190·1782	190·1873	190·1787
ω_4	218·7927	218·7992	218·7931
ω_5	248·6537	248·6714	248·6542

TABLE 6

The first five natural frequencies of a simply supported, uniform Euler–Bernoulli beam carrying five sprung masses. The system parameters are identical to those of Table 5

Nat. Freq. (rad/s)	Reference [11]	Present ($N = 5$)	Present ($N = 14$)
ω_1	150·9571	150·9642	150·9575
ω_2	169·4729	169·4831	169·4733
ω_3	187·9147	187·9275	187·9151
ω_4	217·1279	217·1365	217·1283
ω_5	247·9868	247·9964	247·9872

TABLE 7

The first five natural frequencies of the system of Figure 2, *for* $N = 14$, $x_1 = 0{\cdot}3L$, $k_1 = 12{\cdot}0EI/L^3$; $x_2 = 0{\cdot}6\rho L$, $m_2 = 2{\cdot}0\rho L$; $x_3 = 0{\cdot}8L$, $k_3 = 10{\cdot}0EI/L^3$. *The system parameters are* $EI/L^3 = 6{\cdot}34761 \times 10^4$ N/m *and* $\rho L = 15{\cdot}3875$ kg

Nat. Freq. (rad/s)	Approach I	Approach II
ω_1	223·4154	223·4154
ω_2	1011·2493	1011·2249
ω_3	3460·3277	3460·1041
ω_4	7262·8276	7261·8118
ω_5	10589·6520	10584·3954

$k_1 = 12{\cdot}0EI/L^3$; $x_2 = 0{\cdot}6L$, $m_2 = 2{\cdot}0\rho L$; $x_3 = 0{\cdot}8L$, $k_3 = 10{\cdot}0EI/L^3$. Because the attachment locations do not coincide with the nodes of any component modes, the natural frequencies of Figure 2 can be readily obtained by using equation (16). Using equation (16) as basis, two possible solution schemes are possible. In the first approach, the expressions for σ_i of equation (12) can be modified in accordance with either equation (17) or equation (18), depending on whether the mass is rigidly attached or the spring is grounded respectively. Alternatively, equation (12) can be used directly as it is, but simply let $m_1 \to \infty$ and $m_3 \to \infty$ to reflect that springs k_1 and k_3 are grounded, and let $k_2 \to \infty$ to reflect that mass m_2 is rigidly attached to the beam. Table 7 compares the natural frequencies of the two approaches. For the second scheme, because it is not possible numerically to let the system

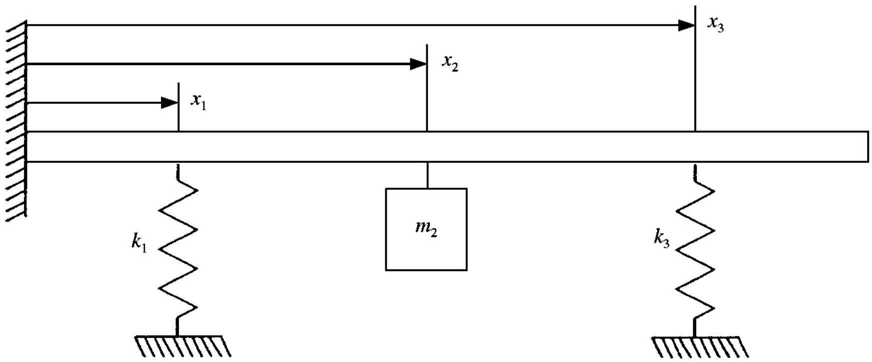

Figure 2. Uniform fixed–free Euler–Bernoulli beam with two grounded springs and one rigidly attached lumped mass.

parameters approach infinity, they are chosen such that $m_1 = m_3 = 1 \times 10^4 \rho L$ and $k_2 = 1 \times 10^4 EI/L^3$. From Table 7, note the excellent agreement between the two methods. While both schemes can be used to find the approximate natural frequencies of the system of Figure 2, the second approach is recommended because the σ_i of equation (12) need not be changed.

4. CONCLUSIONS

An alternative formulation is proposed that can be used to determine the natural frequencies of a linear elastic structure carrying any number of sprung masses. The proposed scheme leads to several noticeable advantages. Specifically, the proposed approach leads to a reduced characteristic determinant that is simple to code; it can be easily extended to accommodate any linear elastic structure with any boundary conditions; it can be easily modified to analyze a linear structure carrying rigid lumped masses or attached to grounded elastic supports. Numerical experiments were performed to validate the proposed approach, and excellent agreements were found between the proposed scheme and known solutions published in the literature.

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